

Quantization of Black Holes in the Wheeler-DeWitt Approach

Thorsten Brotz¹

Fakultät für Physik der Universität Freiburg, Hermann-Herder-Straße 3,
D-79104 Freiburg, Germany

Abstract

We discuss black hole quantization in the Wheeler-DeWitt approach. Our consideration is based on a detailed investigation of the canonical formulation of gravity with special considerations of surface terms. Since the phase space of gravity for non-compact spacetimes or spacetimes with boundaries is ill-defined unless one takes boundary degrees of freedom into account, we give a Hamiltonian formulation of the Einstein-Hilbert-action as well as a Hamiltonian formulation of the surface terms. It then is shown how application to black hole spacetimes connects the boundary degrees of freedom with thermodynamical properties of black hole physics. Our treatment of the surface terms thereby naturally leads to the Nernst theorem. Moreover, it will produce insights into correlations between the Lorentzian and the Euclidean theory. Next we discuss quantization, which we perform in a standard manner. It is shown how the thermodynamical properties can be rediscovered from the quantum equations by a WKB like approximation scheme. Back reaction is treated by going beyond the first order approximation. We end our discussion by a rigorous investigation of the so-called BTZ solution in $2 + 1$ dimensional gravity.

Submitted to Physical Review D

¹e-mail: brotz@physik.uni-freiburg.de

1 Introduction

Most papers about canonical quantum gravity are dealing with spatially compact systems without boundaries. This allows one to skip boundary terms that would otherwise arise from the variation of the Einstein-Hilbert action. On the other hand, the importance of these surface terms was already emphasized by DeWitt [1] and exhaustively discussed for spatially asymptotic spacetimes in [2, 3]. If the spacetime is, for example, not compact, one must take care of the definition of phase space. In general relativity a correct definition can be achieved only if one adds appropriate boundary integrals to the Einstein-Hilbert action. A correct definition of the classical phase space is, of course, a crucial step for the canonical quantization procedure. Thus, a canonical formulation of quantum gravity for spacetimes with boundaries or non-compact spatial regions, e.g. black holes, must also include a discussion of the corresponding surface terms. In the spherically symmetric sector of general relativity this was recently illustrated by Kuchař [4]. In [4] Kuchař explicitly shows, that the reduction of the parametrized action for primordial Schwarzschild black holes to true dynamical degrees of freedom is only achieved if boundary terms (in this case the ADM-mass at infinity) are taken into account.

As is well known, surface terms play also a key role in black hole thermodynamics. This was first discussed by Gibbons and Hawking [5] who could derive the thermodynamical properties of black holes by a path integral formulation of quantum gravity. They found that in the saddle point approximation the thermodynamical laws of black holes are related to surface terms of the Einstein-Hilbert action. A variety of investigations, exploring this relation from distinct viewpoints, led to a deeper understanding of black hole thermodynamics (see for example [6, 7, 8, 9, 10, 11] and references therein). Moreover, a statistical interpretation can be derived in 2+1 dimensional gravity [12].

In recent years another kind of (semiclassical) approximation scheme has been developed, which should also offer an alternative access to a thermodynamical description of black hole mechanics. This scheme is formulated in the Wheeler-DeWitt approach to quantum gravity and is strongly related to a Born-Oppenheimer kind of approximation. It is formally given by a WKB-like expansion of the state functional Ψ [13]. One big advantage of this WKB approximation is that it connects the ‘ill-defined’ quantum equations (Wheeler-DeWitt-equation) with the much better understood quantum field theory in curved spacetimes. Moreover, it provides the theory of quantum gravity with a semiclassical time function called WKB-time. This WKB-time is tied to the recovery of a functional Schrödinger equation for matter fields on a curved background from the Wheeler-DeWitt equation. Thus, *time* has to be viewed in this approach as a semiclassical property. Consequently, the theory of quantum gravity must be interpreted as ‘timeless’ on the full level. Until now only little attention has been paid to boundary terms in formulating the WKB approximation. On the

other hand, it was emphasised recently that for the recovery of black hole thermodynamics from the WKB approximation boundary terms have to be included [8].

In this paper we shall give a constrained formulation of the ‘boundary states’ and illustrate how this leads to a consistent canonical quantum formulation. We will concentrate on the case of ‘black hole boundaries’, since we are mainly interested in the discussion of thermodynamical properties. Extension to other boundaries can easily be achieved. The study will generalize the results of [8], where only spherically symmetric black holes were discussed. Our classical description will uncover relations of the boost parameter at the bifurcation point and the opening angle of the Euclidean spacetime with the surface gravity of black holes. Moreover, the Nernst theorem for black hole mechanics will appear as a simple consequence of the obtained relations. For quantization the used constrained formulation of the boundary terms will prove to be very advantageous. Nevertheless, interpretational problems still appear. These problems are avoided in the above mentioned WKB approximation. The classical thermodynamical description can be found in the highest order of this approximation scheme. Moreover, we will demonstrate how back reaction influences the entropy of black holes.

Our paper is organized as follows. In section 2 we give a brief review of the Hamiltonian formulation of the Einstein-Hilbert action for compact spacetimes \mathcal{M} with topology $\Sigma \times \mathbb{R}$ and boundary $\partial\mathcal{M}$ following, with elaborations, the work of [14]. Beyond that we explicitly discuss the Hamiltonian formulation of the surface terms, since this is crucial for our further investigation. In section 3 the application of section 2 for the particular case of black hole spacetimes is discussed. We shall demand \mathcal{M} to lie within the right wedge of the corresponding Kruskal diagram, and its boundary $\partial\mathcal{M}$ to include the bifurcation two-sphere as well as spatial infinity, where we assume our theory to be Poincaré invariant. Section 4 is devoted to thermodynamical considerations that will relate our Hamiltonian treatment of the surface terms with the standard results found by other methods. It includes an explicit computation of the boundary term at the bifurcation point. This computation will connect the boost parameter at the bifurcation point [15] with the surface gravity of the black hole. Next we show the relation between the boundary degrees of freedom and the first law of black hole mechanics by using perturbational methods discussed in [16]. Moreover, we give a proof of the Nernst theorem for black holes, which was for a long time thought to be violated [17]. Since a similar proof of the Nernst theorem was found in the Euclideanized theory of gravity recently [7, 18, 19], we will relate in section 5 our results with those of the Euclidean approach. We also comment on topological considerations in the realm of the Lorentzian theory which were recently discussed to explain the meaning of entropy for gravitating systems [8, 20]. In section 6 we pass over to the quantum theory by Dirac’s constrained quantization method. In contrast to the standard midisuperspace quantization this will

now include also boundary constraints.² Since it is still unclear how to extract thermodynamical properties from the full quantum equations we shall overcome this problem in section 7 by giving an appropriate semiclassical approximation. This semiclassical approximation scheme connects the WKB-phase of the wavefunction with the classical laws of black hole mechanics. Going beyond the first order of approximation leads to back reaction corrections. In section 8 we show how our treatment works in the particular example of $2 + 1$ gravity. We start with a purely classical description of the so called BTZ black hole. This will include a derivation of the Hamilton-Jacobi functional. Then the case of conformally coupled matter fields on the background of the BTZ solution is discussed. Moreover, it is shown how back reaction leads to corrections of entropy and temperature. The derived entropy, which follows directly from the deduced boundary terms, differs from the one given in [21]. The reason for this difference is found to follow from contributions of the conformal coupling (which are included in our treatment, but not in [21]). These contributions are important, since otherwise the first thermodynamical law for black holes $dM = TdS$ does not hold. Finally we shall briefly comment on results in string theory, where recently a statistical interpretation of black hole thermodynamics was given (for a review see e.g. [22]), and where a different conclusion has been drawn about the behaviour of the entropy in the extremal cases.

2 Action and ADM Decomposition

Let \mathcal{M} denote a four manifold of topology $\Sigma \times \mathbb{R}$ endowed with a Lorentzian metric $g_{\mu\nu}$. The derivative operator associated with $g_{\mu\nu}$ is called ∇_μ . The boundary of \mathcal{M} consists of initial and final spacelike hypersurfaces Σ_0 and Σ_1 , and a timelike hypersurface $B = B_t \times \mathbb{R}$ [Fig. 1]. The induced metric on the spacelike hypersurface Σ is denoted by h_{ab} , the induced metric on the timelike boundary B by γ_{ij} , and the induced metric on the edges B_0 and B_1 by σ_{ab} . The Einstein-Hilbert action of general relativity is in this case given by [15]:

$$I[\mathcal{M}, g] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} (\mathcal{R} - 2\Lambda) + \frac{1}{8\pi G} \int_{\Sigma_0}^{\Sigma_1} d^3x \sqrt{h} K \quad (1) \\ + \frac{1}{8\pi G} \int_B d^3x \sqrt{-\gamma} \Theta + \frac{1}{8\pi G} \int_{B_0}^{B_1} d^2x \sqrt{\sigma} \sinh^{-1} \eta,$$

where Λ is the cosmological constant. The integral between $\Sigma_0(B_0)$ and $\Sigma_1(B_1)$ is an abbreviation for the integral over the hypersurface $\Sigma_1(B_1)$ minus the integral over the hypersurface $\Sigma_0(B_0)$. K denotes the trace of the extrinsic curvature K_{ab} of the boundaries Σ_0 and Σ_1 , and Θ the trace of the extrinsic curvature Θ_{ij} for the boundary B . The final term of our action (1) has to be included as a joint

²For a review of the state of art in the midisuperspace quantization of black holes see [4].

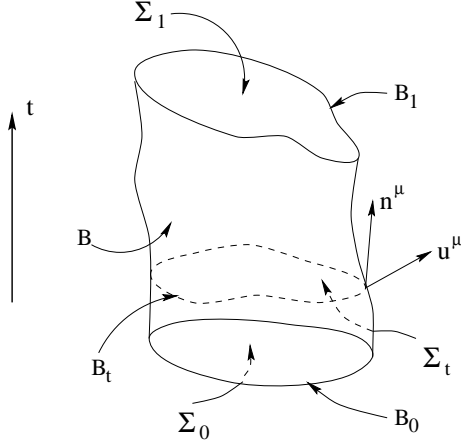


Figure 1. A spacetime manifold M with its boundaries Σ and B is shown. The 3+1 decomposition of the manifold is depicted. One dimension is suppressed.

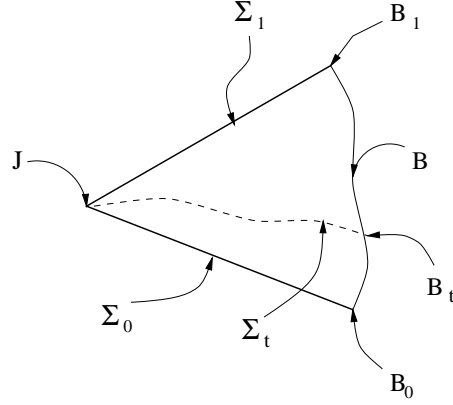


Figure 2. A spacetime diagram showing the disconnected joint J at the bifurcation point and the outer boundary B . Two spacetime dimensions are suppressed.

correction term since we are dealing with spacetimes with non-smooth boundaries denoted by B_0 and B_1 [15]. Thereby

$$\eta = n_\mu u^\mu \quad (2)$$

measures the non-orthogonality of the unit normals n^μ to Σ_t and u^μ to B . The ambiguity in the orientation of n^μ and u^μ is fixed by requiring that they will be future pointing and outward pointing, respectively.

The action (1) is only well defined for spatially compact geometries but is divergent for noncompact ones. In the latter case one first has to fix a reference background (\mathcal{M}, g_0) and then to consider the action $I_{reg}[\mathcal{M}, g] = I[\mathcal{M}, g] - I[\mathcal{M}, g_0]$ [19].

For the ADM decomposition we write the line element of \mathcal{M} in the standard form

$$ds^2 = -(N^2 - h_{ab}N^aN^b)dt^2 + 2h_{ab}N^adx^bdt + h_{ab}dx^adx^b, \quad (3)$$

where N and N^a denote as usual the lapse function and the shift vector. The scalar curvature \mathcal{R} can be rewritten by use of the Gauss-Codazzi relations as

$$\mathcal{R} = R + K_{ab}K^{ab} - K^2 + 2\nabla_\mu(n^\mu K - a^\mu), \quad (4)$$

where $a^\mu = n^\nu \nabla_\nu n^\mu$ is the acceleration of the unit normal n^μ , and R is the scalar curvature belonging to the hypersurface Σ_t with induced metric h_{ab} . After inserting relation (4) into the action (1) we can convert the integral over the total divergence in (4) into a surface integral over the boundary of \mathcal{M} . Thus, the action reads

$$I[\mathcal{M}, g] = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} [R + K_{ab}K^{ab} - K^2 - 2\Lambda] \quad (5)$$

$$\begin{aligned}
& + \frac{1}{8\pi G} \int_B d^3x \sqrt{-\gamma} [\Theta + u_\mu (n^\mu K - a^\mu)] \\
& + \frac{1}{8\pi G} \int_{B_0}^{B_1} d^2x \sqrt{\sigma} \sinh^{-1} \eta.
\end{aligned}$$

Legendre transformation then leads to the standard Hamiltonian formulation of the action

$$\begin{aligned}
I[\mathcal{M}, g] &= \int dt \int_{\Sigma_t} d^3x [p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a - 2D_a(p^{ab} N_b)] \\
& + \frac{1}{8\pi G} \int_B d^3x \sqrt{-\gamma} [\Theta + u_\mu (n^\mu K - a^\mu)] \\
& + \frac{1}{8\pi G} \int_{B_0}^{B_1} d^2x \sqrt{\sigma} \sinh^{-1} \eta,
\end{aligned} \tag{6}$$

with the Hamilton constraint

$$\mathcal{H} \equiv \frac{8\pi G}{\sqrt{h}} (2p_{ab} p^{ab} - p^2) - \frac{\sqrt{h}}{16\pi G} (R - 2\Lambda) \approx 0, \tag{7}$$

and with the diffeomorphism constraints

$$\mathcal{H}^a \equiv -2D_b p^{ab} \approx 0, \tag{8}$$

where $p^{ab} = \sqrt{h}(K^{ab} - Kh^{ab})/16\pi G$ are the canonical momenta conjugate to h_{ab} , and D_a denotes the derivative operator associated with the induced metric h_{ab} . Note that the action (6) is still not written in canonical form, since the boundary terms do not possess a canonical form. Following the computation in [14] we can write

$$\begin{aligned}
I[\mathcal{M}, g] &= \int dt \int_{\Sigma_t} d^3x [p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a] \\
& + \frac{1}{8\pi G} \int dt \int_{B_t} d^2x N \sqrt{\sigma} [k + \omega \sinh^{-1} \eta (\nabla_\mu v^\mu)] \\
& - 2 \int dt \int_{B_t} d^2x r_a p_\sigma^{ab} N_b,
\end{aligned} \tag{9}$$

with $\omega = (1 + \eta^2)^{-1/2}$, with the projections $r_\mu = \omega(u_\mu + \eta n_\mu)$ of u_μ onto Σ_t and $v_\mu = \omega(n_\mu - \eta u_\mu)$ of n_μ onto B , with k denoting the trace of the extrinsic curvature for the boundary B_t , and with the canonical momenta $p_\sigma^{ab} = \sqrt{\sigma}(K^{ab} - Kh^{ab})/16\pi G$ evaluated on the boundary B_t .

Note that the variation of the lapse function and shift vector on the boundary leads to the unwanted conclusion

$$\sqrt{\sigma} [k + \omega \sinh^{-1} \eta (\nabla_\mu v^\mu)] = 0, \tag{10}$$

$$r_a p_\sigma^{ab} = 0. \tag{11}$$

This inconsistency forces one to demand that the values of the lapse function and shift vector cannot be varied at the boundary. This corresponds to a gauge fixation at the boundary. To get rid of this unsatisfying gauge fixation one can perform a parametrization of lapse and shift [4]

$$N|_{B_t}(t) = \dot{\tau}(t), \quad (12)$$

$$N_a|_{B_t}(t) = \dot{\tau}_a(t), \quad (13)$$

and treat in the following τ and τ_a as additional ‘boundary’ variables. Variation of these variables gives then the conservation laws

$$\frac{d}{dt}\sqrt{\sigma} \left[k + \omega \sinh^{-1} \eta (\nabla_\mu v^\mu) \right] = 0, \quad (14)$$

$$\frac{d}{dt} r_a p_\sigma^{ab} = 0. \quad (15)$$

We may now view (9) as being written in a mixed Hamiltonian and Lagrangian form. In order to find the Hamiltonian formulation also for the boundary terms one has to define the canonical conjugate momenta π and π^a by a standard Legendre transformation. But this can only be consistently done if new constraints are introduced:

$$\mathcal{C} \equiv \pi - \frac{1}{8\pi G} \sqrt{\sigma} \left[k + \omega \sinh^{-1} \eta (\nabla_\mu v^\mu) \right] \approx 0, \quad (16)$$

$$\mathcal{C}^a \equiv \pi^a + 2r_b p_\sigma^{ab} \approx 0. \quad (17)$$

These constraints must be adjoined to the action by Lagrange multipliers $N|_{B_t}$ and $N_a|_{B_t}$:

$$\begin{aligned} I[\mathcal{M}, g] &= \int dt \int_{\Sigma_t} d^3x \left[p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a \right] \\ &\quad + \int dt \int_{B_t} d^2x \left[\pi \dot{\tau} + \pi^a \dot{\tau}_a - N\mathcal{C} - N_a \mathcal{C}^a \right]. \end{aligned} \quad (18)$$

For simplicity we will introduce the notion of ‘outer constraint’ for constraints existing only on the boundaries and ‘inner constraint’ for all the others. Moreover, we shall denote the inner constraints by \mathcal{H} , while for the outer constraints we shall write \mathcal{C} .

In the noncompact case one has also to take into account a background action (as mentioned above). In this case the Hamiltonian is given by the difference between the above computed Hamiltonian and the one computed for the background.

3 Application to Black Holes

In the last section we have succeeded in formulating a canonical theory that also takes into account additional degrees of freedom related to non-vanishing boundary terms of the Einstein-Hilbert action. But we are still not at the point where we can treat the particular case of black holes. Until now we have only examined joints which are embedded in three-dimensional surfaces, whereas for black holes we have to consider isolated joints. These isolated joints are related to hypersurfaces beginning at the bifurcation point. Therefore, they are disconnected from the boundary belonging to the spatially asymptotic flat region [Fig. 2]. The Einstein-Hilbert action of general relativity is in this case given by [15]:

$$\begin{aligned} I[\mathcal{M}, g] = & \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} (\mathcal{R} - 2\Lambda) + \frac{1}{8\pi G} \int_{\Sigma_0}^{\Sigma_1} d^3x \sqrt{h} K \\ & + \frac{1}{8\pi G} \int_B d^3x \sqrt{-\gamma} \Theta + \frac{1}{8\pi G} \int_{B_0}^{B_1} d^2x \sqrt{\sigma} \sinh^{-1} \eta \\ & + \frac{1}{8\pi G} \int_J d^2x \sqrt{\sigma} \sinh^{-1} \eta, \end{aligned} \quad (19)$$

where the last term takes into account the isolated joint. Performing now the ADM decomposition as demonstrated in the last section, the action (19) transforms into the expression

$$\begin{aligned} I[\mathcal{M}, g] = & \int dt \int_{\Sigma_t} d^3x \left[p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a \right] \\ & + \int dt \int_{B_t} d^2x \left[\pi \dot{\tau} + \pi^a \dot{\tau}_a - N\mathcal{C} - N_a \mathcal{C}^a \right] \\ & + \frac{1}{8\pi G} \int_J d^2x \sqrt{\sigma} \sinh^{-1} \eta - 2 \int dt \int_J d^2x r_n (p_\sigma^{mn} N_m). \end{aligned} \quad (20)$$

Note that in order to fix the hypersurfaces Σ_t at the bifurcation point we have to demand

$$N|_J = 0, \quad N_r|_J = 0. \quad (21)$$

The vanishing of the lapse function is thereby responsible for the cancellation of the surface terms belonging to the total derivative in (4) evaluated for Σ_0 and Σ_1 at the joint J . On the other hand, not all diffeomorphisms are forbidden at the bifurcation point (only those which do not generate diffeomorphisms of the bifurcation two-sphere) [7]. The presence of these non-vanishing diffeomorphisms explains the appearance of the last term in (20). To transcribe also the boundary terms at J into canonical form we write

$$\int_J d^2x \sqrt{\sigma} \sinh^{-1} \eta = A(r_+) \int_{t_0}^{t_1} dt \frac{d}{dt} \sinh^{-1} \eta(t) = \int dt A(r_+) \dot{\theta} \quad (22)$$

with the horizon area $A(r_+) = \int_J d^2x \sqrt{\sigma}$ and boost parameter $\theta = \sinh^{-1} \eta$.³ Note that $\eta(t_0) = 0$ and $\eta(t_1) = \eta$, since $\eta(t)$ describes the angle between the hypersurfaces Σ_0 and Σ_1 . For simplicity we write

$$\pi_J^m = -2r_n p_\sigma^{mn}|_J. \quad (23)$$

Performing now the described parametrization at the boundary J we end with the canonical action

$$\begin{aligned} I[\mathcal{M}, g] = & \int dt \int_{\Sigma_t} d^3x \left[p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a \right] \\ & + \int dt \int_{B_t} d^2x \left[\pi \dot{\tau} + \pi^a \dot{\tau}_a - N\mathcal{C} - N_a \mathcal{C}^a \right] \\ & + \int dt \left[\pi_\theta \dot{\theta} - \kappa \mathcal{C}_J + \int_J d^2x \left(\pi_J^m \dot{\theta}_m - N_m \mathcal{C}_J^m \right) \right] \end{aligned} \quad (24)$$

with additional constraints

$$\mathcal{C}_J \equiv \pi_\theta - \frac{A(r_+)}{8\pi G} \approx 0, \quad (25)$$

$$\mathcal{C}_J^m \equiv \pi_J^m + 2r_n p_\sigma^{mn} \approx 0. \quad (26)$$

Note that $\kappa \equiv \dot{\theta} = \frac{d}{dt} \sinh^{-1} \eta$ is not connected with the lapse function $N|_J$. Since we are dealing with black holes, we assume covariance under Poincaré transformations at infinity. Such a Hamiltonian formulation was presented by Regge and Teitelboim [2]. Thus we will take over their description and transform it to the above described constrained formulation. As in their derivation we write for the lapse function and the shift vector at infinity

$$N = \alpha^0 + \beta^0_l x^l, \quad (27)$$

$$N^a = \alpha^a + \beta^a_l x^l, \quad (28)$$

where $\{x^l\}$ are asymptotically cartesian coordinates and $\beta_{ab} = -\beta_{ba}$. This leads to a split in the surface term

$$N_a \pi^a = -2 \int_{B_t} d^2x r_a N_b p_\sigma^{ab} = \alpha_a P^a - \frac{1}{2} \beta_{ab} M^{ab}, \quad (29)$$

where

$$P^a = -2 \int_{B_t} d^2x (r_b p_\sigma^{ab}), \quad (30)$$

$$M^{ab} = -2 \int_{B_t} d^2x r_c (x^a p_\sigma^{bc} - x^b p_\sigma^{ac}). \quad (31)$$

³In Euclidean spacetimes this boost corresponds to a rotation and is thus also called deficit or opening angle.

Note that all components of α and β are functions of t only. In addition we define

$$P^0 = \frac{1}{8\pi G} \int_{B_t} d^2x \sqrt{\sigma} (k - k_0) = -M, \quad (32)$$

where we have used the result of [19] that the integral over $k_{reg} = k - k_0$ can be identified with the negative ADM-energy M . Let us for simplicity assume here and in the following $\eta = 0$ at $B_\infty = B_t$, then $P^0 = \int_{B_t} d^2x \pi$. The next step is to parametrize this new form of the surface terms considering P^0 , P^a and M^{ab} as new momenta. This leads to the action

$$\begin{aligned} I[\mathcal{M}, g] = & \int dt \int_{\Sigma_t} d^3x \left[p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a \right] \\ & + \int dt \left[P^0 \dot{\tau}_0 + P^a \dot{\tau}_a + M^{ab} \dot{\tau}_{ab} - \alpha_0 \mathcal{C}^0 - \alpha_a \mathcal{C}^a - \beta_{ab} \mathcal{C}^{ab} \right] \\ & + \int dt \left[\pi_\theta \dot{\theta} - \kappa \mathcal{C}_J + \int_J d^2x \left(\pi_J^m \dot{\theta}_m - N_m \mathcal{C}_J^m \right) \right] \end{aligned} \quad (33)$$

where we have introduced the constraints

$$\mathcal{C}^0 \equiv P^0 + M \approx 0, \quad (34)$$

$$\mathcal{C}^a \equiv P^a + 2 \int_{B_t} d^2x \left[r_b p_\sigma^{ab} \right] \approx 0, \quad (35)$$

$$\mathcal{C}^{ab} \equiv -\frac{1}{2} M^{ab} - \int_{B_t} d^2x r_c \left[x^a p_\sigma^{bc} - x^b p_\sigma^{ac} \right] \approx 0. \quad (36)$$

The action (33) with the constraints (7), (8), (25), (26), (34), (35) and (36) presents our main result of this section. It will be used below to discuss thermodynamics of black holes as well as to formulate the quantized theory. To be explicit, we will write down also the Hamiltonian

$$\begin{aligned} H = & \int_{\Sigma_t} [N\mathcal{H} + N_a \mathcal{H}^a] d^3x + \int_J N_m \mathcal{C}_J^m d^2x \\ & + \kappa \mathcal{C}_J + \alpha_0 \mathcal{C}^0 + \alpha_a \mathcal{C}^a + \beta_{ab} \mathcal{C}^{ab} \end{aligned} \quad (37)$$

which can, of course, be immediately read off from (33).

4 Thermodynamics and Boundary States

In this section we show how thermodynamics can be extracted from the Hamiltonian (37). To hold our discussion as transparent as possible we shall restrict our consideration to the particular case of Kerr black holes. Therefore we assume $N^\mu \equiv (N^0, N^a)$ to asymptotically approach a linear combination of time translation and rotation (with angular velocity Ω) at infinity:

$$N^0 \rightarrow N_\infty = 1, \quad N^a \rightarrow \Omega \epsilon^a_{bc} \varphi^b x^c. \quad (38)$$

This means $\alpha^0 = N_\infty$, $\alpha^a = 0$, and $\beta_{ab} = -\Omega\epsilon_{abc}\varphi^c$. The particular form of β_{ab} contains

$$\frac{1}{2}\beta_{ab}M^{ab} = \Omega \int_{B_t} d^2x r_c \epsilon_{abd} \varphi^d (x^a p_\sigma^{bc} - x^b p_\sigma^{ac}) = -\Omega J, \quad (39)$$

where

$$J = \varphi^d L_d = 2\varphi^d \int_{B_t} d^2x r_c \epsilon_{dba} x^a p_\sigma^{bc} \quad (40)$$

is the total angular momentum [2]. Thus (33) can be reduced to

$$\begin{aligned} I^{Kerr}[M, g] = & \int dt \int_{\Sigma_t} d^3x [p^{ab} \dot{h}_{ab} - N\mathcal{H} - N_a \mathcal{H}^a] \\ & \int dt [P^0 \dot{\tau}_0 + P^\omega \dot{\tau}_\omega + \pi_\theta \dot{\theta} - N_\infty \mathcal{C}_0 - \Omega \mathcal{C}_\Omega - \kappa \mathcal{C}_J] \end{aligned} \quad (41)$$

with

$$\mathcal{C}_\Omega \equiv -P^\omega + J \approx 0. \quad (42)$$

Let us next comment on the variation of (41). The variation with respect to the variables p^{ab} and h_{ab} leads to the equations of motion and a set of boundary terms which are cancelled by the variation of M , J , and A . The variation of the various Lagrange multipliers produces the constraints \mathcal{H}_{inner} and \mathcal{C}_{outer} . The conservation laws $\dot{M} = 0$, $\dot{J} = 0$, and $\dot{A} = 0$ follow from $\delta\dot{\tau}_0$, $\delta\dot{\tau}_\omega$, and $\delta\dot{\theta}$, respectively. Finally, the variation of the boundary momenta P^0 , P^ω , and π_θ provides the relations $\dot{\tau}_0 = N_\infty$, $\dot{\tau}_\omega = -\Omega$, and $\dot{\theta} = \kappa$. While the meaning of the Lagrange multipliers N_∞ and Ω is well-understood, the meaning of κ remains unclear. One is thus left with the question what quantity is related to the change of the boost parameter θ . An answer to this question can be found from the boundary term of the bifurcation point and requires a computation of the extrinsic curvature. To perform this calculation it is convenient to use the relation

$$\Theta = -\nabla_\alpha r^\alpha, \quad (43)$$

where r^α denotes the outward pointing unit normal to the boundary of the hypersurface Σ at the bifurcation point. Note that the normal r^α lies in the tangent space of Σ and is pointing by definition in negative r -direction. This explains the minus sign on the right hand side of equation (43). Since the evaluation of the boundary term at the bifurcation point cannot depend on the foliation between the initial (Σ_0) and final (Σ_1) hypersurfaces we assume for the following that near the bifurcation point the time flow vector field is given by the Killing vector field χ , which is associated with any bifurcation horizon. The lapse function N and the shift vector N^a thus satisfy

$$\chi = Nn + \vec{N}, \quad (44)$$

where n is the unit normal vector of the hypersurfaces Σ . Since χ vanishes at the bifurcation surface, (44) implies

$$N|_J = 0, \quad N^a|_J = 0. \quad (45)$$

Moreover, it can be shown [23]

$$\left. \frac{N^a}{(-\chi^\alpha \chi_\alpha)} \right|_J = 0, \quad \left. \frac{N^a}{N} \right|_J = 0 \quad (46)$$

and

$$\nabla_\alpha N|_J = -\kappa r_\alpha, \quad (47)$$

where κ denotes the surface gravity. Thus, when we approach the bifurcation point from an arbitrary hypersurface Σ , the shift vector vanishes more rapidly than the lapse function. Therefore, χ becomes orthogonal to Σ at the bifurcation surface, so that

$$\chi^\alpha r_\alpha|_J = 0. \quad (48)$$

As $(-\chi^\alpha \chi_\alpha)^{1/2} = \sqrt{N^2 - N^a N_a}$, the gradient of $(-\chi^\alpha \chi_\alpha)^{1/2}$ is given by

$$\nabla_\alpha (-\chi^\beta \chi_\beta)^{1/2} = \frac{N \nabla_\alpha N}{(-\chi^\alpha \chi_\alpha)^{1/2}} - \frac{N^b \nabla_\alpha N_b}{(-\chi^\alpha \chi_\alpha)^{1/2}}, \quad (49)$$

where the second term vanishes at the horizon because of (46). On the horizon one has in addition the relation $\nabla_\alpha (\chi^\beta \chi_\beta) = -2\kappa \chi_\alpha$ between χ and the surface gravity κ [16]. Using (46) as well as $(-\chi^\alpha \chi_\alpha)|_J = N^2$, and taking into account equation (47), one can write for (49)

$$\frac{1}{N} \chi_\alpha = -r_\alpha. \quad (50)$$

Note that the last equation only holds on the horizon. Thus we cannot differentiate this equation by applying ∇^α . But we can apply $\chi^{[\gamma} \nabla^{\alpha]}$ to any equation holding on the horizon [17]. Differentiating (50) with $\chi^{[\gamma} \nabla^{\alpha]}$ and contracting afterwards with χ_γ leads to

$$-\nabla^\alpha r_\alpha = \frac{\kappa}{N}, \quad (51)$$

where also use of the Killing property $\nabla_{(\alpha} \chi_{\beta)} = 0$ and the relation $r^\alpha \nabla_\gamma r_\alpha = 0$ was made. As expected, the trace of the extrinsic curvature is divergent at the horizon. Nevertheless, inserting (51) into the boundary term one gets a finite result,

$$\int_{S^2 \times I} d^3x \, \Theta \sqrt{-\gamma} = \int_{S^2 \times I} d^3x \, \kappa \frac{1}{N} \sqrt{-\gamma} = \int dt \, \kappa A(r_+), \quad (52)$$

where the last equality follows from the line element (3) by use of (46). Comparison of the expressions (22) and (52) thus yields the relation

$$\dot{\theta} = \kappa, \quad (53)$$

which we have anticipated above with the used notation and which is the answer to our foregoing question. We see that the boost parameter connecting the unit

normals between two hypersurfaces Σ_{t_1} and Σ_{t_2} is given by $\theta = \kappa(t_2 - t_1)$. Since the surface gravity κ is zero for extreme black holes, also the surface term at the bifurcation point is vanishing in these cases. Thus, in the extreme cases the boundary term at the bifurcation point produces no additional degree of freedom. As this additional degree of freedom is connected with the entropy of black holes [8], one may suspect that extreme black holes have zero entropy as well as zero temperature. This point will be discussed further in the next section.

The next goal we shall strive for is the derivation of the first law of black hole mechanics. This goal can be achieved by a slight modification of the perturbation procedure described in [11, 16]. Let $[h_{ab}, p^{ab}; \tau_0, \tau_\omega, \theta, P^0, P^\omega, \pi_\theta]$ be any initial asymptotically flat data set satisfying the constraints and let $[\delta h_{ab}, \delta p^{ab}; \delta \tau_0, \delta \tau_\omega, \delta \theta, \delta P^0, \delta P^\omega, \delta \pi_\theta]$ be any smooth asymptotically flat perturbation satisfying the linearized constraints. If the Hamiltonian is a linear combination of constraints, it is obvious that for stationary spacetimes δH must vanish. For our Hamiltonian this means

$$\frac{\kappa}{8\pi G} \delta \pi_\theta + N_\infty \delta P^0 + \Omega \delta P^\omega = 0. \quad (54)$$

Note that the perturbation of h_{ab}, π^{ab} implies a perturbation of M , A , and J . Use of the constraints transforms (54) into the standard relation

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega \delta J. \quad (55)$$

For the more general form of the Hamiltonian (37) the same consideration leads to

$$\kappa \delta \pi_\theta + \alpha_0 \delta P^0 + \alpha_a \delta P^a - \frac{1}{2} \beta_{ab} \delta P^{ab} + \int_J N_m \delta \pi_J^m d^2 x = 0. \quad (56)$$

We have thus found a generalization of the standard formula (55) (here given for spacetimes which are Poincaré invariant). For a discussion of the last term in (56) see [7].

The equation (55) (or more general (56)) provides the theory with the so called first law of black hole mechanics. The identification of $\kappa/2\pi$ with the temperature and $A/4G$ with the entropy of black holes is not achieved by these means. This still relies on the quantum field theory scattering calculation of Hawking [24] or the path integral formulation given by Gibbons and Hawking [5].

5 Entropy, Nernst Theorem and Topology

In this section we shall reformulate the treatment of [5] to define the entropy of black holes from the above discussed boundary states. This will also explain how the vanishing of the boundary term is related to the zero entropy interpretation. The central point of the following investigation is the action evaluated for stationary spacetimes ($\dot{h}_{ab} = 0, \dot{p}^{ab} = 0$), which in a gravitating system must be

viewed to describe equilibrium states. On-shell the constraints are fulfilled and thus (41) reduces to

$$I = \int dt \left[\frac{\kappa}{8\pi G} A(r_+) - M + \Omega J \right]. \quad (57)$$

The surface gravity κ was found to be [24]

$$\kappa = 2\pi\beta^{-1}, \quad (58)$$

where β denotes the inverse temperature at infinity. For the definition of entropy one can now refer to the path integral formulation. The connection between the path integral approach and thermodynamics is obtained from the transition amplitude $\langle q_f t_f | q_i t_i \rangle = \int Dq \exp(iI[q]/\hbar)$, which in the Schrödinger picture can also be written as $\langle q_f | \exp[-i\hat{H}(t_f - t_i)/\hbar] | q_i \rangle$, where H denotes the Hamiltonian. In the latter case the partition function can be expressed by putting $q_f = q_i$, summation over a complete set of eigenstates, and identification of $t_f - t_i$ with $-i\hbar\beta = -i\hbar/k_B T$ (i.e. $Z = \text{tr}(\exp[-\hat{H}/k_B T])$). Applying the same procedure to the path integral expression of the transition amplitude leads to

$$Z = \int Dq e^{-I_E/\hbar}, \quad (59)$$

where I_E is the Euclidean action resulting from a Wick rotation $t \rightarrow i\tau$, and where the integral is over all q 's which are periodic in τ with periodicity $\hbar\beta$. In the saddle point approximation the Euclidean action for (41) can be easily read off from (57) ($\hbar = 1$):

$$I^E = \beta \frac{\kappa}{8\pi G} A - \beta M + \beta \Omega J. \quad (60)$$

For extremal black holes the same consideration will lead to a Euclidean action

$$I_{ext}^E = -\beta M + \beta \Omega J. \quad (61)$$

Moreover, in the saddle point approximation (because of (59)) the Euclidean action I^E is directly related to the thermodynamical potential of a grand canonical ensemble via $I^E = -\beta W$ [5]. The total differential dW calculated from (60) therefore reads

$$dW = -\frac{A}{4G} dT - J d\Omega, \quad (62)$$

where use was made of (55) and where the identification (58) with $\beta = T^{-1}$ was put in ($k_B = 1$). Thus in the case of nonextreme black holes one finds the Bekenstein-Hawking formula for the entropy:

$$S = - \left(\frac{dW}{dT} \right)_{\Omega} = \frac{A}{4G}. \quad (63)$$

In the extreme case one has to use the total derivative of the potential defined by (61) and thus finds

$$S_{ext} = - \left(\frac{dW_{ext}}{dT} \right)_{\Omega} = 0. \quad (64)$$

Hence, in the extreme case one has a vanishing entropy, although the horizon area itself is not vanishing. This is a remarkable result, which follows here from a consistent treatment of the Hamiltonian formulation.

It is also interesting to compare our expression for the thermodynamical potential W with the one found by Gibbons and Hawking [5]. In their approach only one boundary (located at infinity) with topology $S_1 \times S_2$ is considered. Evaluation of the integral over the trace of the extrinsic curvature $K_{reg} = K - K_0$ leads thereby to the potential

$$W = \frac{1}{2}M. \quad (65)$$

The equivalence of the expression (65) with the one discussed above can be easily derived by use of the Smarr formula

$$\frac{1}{2}M = \frac{\kappa}{8\pi G}A + \Omega J. \quad (66)$$

Given (65), the entropy can be deduced from the standard formula

$$S = \left(\beta \frac{\partial}{\partial \beta} - 1 \right)_J I^E. \quad (67)$$

Since the potentials defined by (60) and (65) are equivalent one must assume M to be β -independent ($\partial M / \partial \beta = 0$) in the extreme cases. This ‘missing’ relation between M and β in the extreme cases appearing here from consistency considerations is also observed in [19] and is there made responsible for a vanishing entropy.

Let us next comment on some topological issues discussed in the Euclidean approach and connect them to the results derived above. In the Euclidean theory black hole spacetimes have the topology $\mathbb{R}^2 \times S^{D-2}$. Near the horizon one can take the line element to be given by [7]

$$ds^2 = d\rho^2 + \rho^2 \theta_E^2 d\tau^2 + \gamma_{mn}(dx^m + N^m d\tau)(dx^n + N^n d\tau), \quad (68)$$

where we have chosen a polar system of coordinates for the description of \mathbb{R}^2 . Note that $\rho = 0$ describes here the position of the horizon, θ_E is the proper angle of an arc in \mathbb{R}^2 , and τ denotes the Killing time. Since a boost at the bifurcation point corresponds to a rotation in the Euclidean spacetime, a ‘Wick rotation’ of the boost parameter θ gives the Euclidean opening angle θ_E (explicitly, $\theta_E = i\theta$). Thus we can conclude,

$$\theta_E = \kappa(\tau_2 - \tau_1) = 2\pi\beta^{-1}(\tau_2 - \tau_1). \quad (69)$$

By demanding regularity for the line element (68), we thus find immediately the period β for the τ -coordinate. This nicely displays the correlation between regularity of the Euclidean line element and black hole temperature, as it is used in the path integral approach [5] discussed above. For extreme black holes θ_E vanishes identically and therefore the metric (68) is no longer well-defined. As a way out one can define the line element in the extreme cases to read

$$ds^2 = d\rho^2 + e^{\rho^2} d\tau^2 + \gamma_{mn}(dx^m + N^m d\tau)(dx^n + N^n d\tau). \quad (70)$$

As a consequence of the metric (70) the position of the horizon is now given by $\rho = -\infty$ and belongs no longer to the manifold itself. Hence, extreme black holes have the topology of an annulus and not of a disk (as the nonextreme ones). Since the topology of a classical theory is a quality one tries to preserve under quantization one here must conclude, as discussed in [7, 19], that the entropy of extremal black holes is zero. The relation (69) derived from the Lorentzian theory in the last section now connects the observed topology change directly with the vanishing of the surface gravity κ . Since the temperature of black holes is proportional to the surface gravity, the observed topology change is thus directly related to the vanishing of the temperature.

More recently, another interpretation for the entropy of black holes was drawn which is also connected to topological issues but this time in the Lorentzian theory [8, 20]. By investigations of the Penrose diagram it is quite obvious that for nonextreme black holes the data within call from an arbitrary slice Σ starting at the bifurcation sphere allow not to recover the full spacetime. This drastically differs from the extreme cases, where the maximum information is already contained in such a hypersurface. Also here the change in the Penrose diagrams is caused by the surface gravity κ .

6 Canonical Quantization

For the discussion of the canonical quantum theory of gravity we will also consider interactions with matter fields. For simplicity we shall here represent these fields by a minimally coupled scalar field.⁴ As a consequence the Hamiltonian and diffeomorphism constraints acquire additional contributions \mathcal{H}_ϕ and \mathcal{H}_ϕ^a ,

$$\mathcal{H} \equiv \frac{1}{2\mathcal{M}} G_{abcd} p^{ab} p^{cd} + \mathcal{M}V[h_{ab}] + \mathcal{H}_\phi \approx 0, \quad (71)$$

$$\mathcal{H}^a \equiv -2D_b p^{ab} + \mathcal{H}_\phi^a \approx 0, \quad (72)$$

⁴For a non-minimally coupling – a coupling of the scalar field not only to the metric but also to the curvature \mathcal{R} – we would have to consider additional boundary terms. This we shall discuss in section 8.

where we have for convenience introduced the DeWitt metric

$$G_{abcd} = \frac{1}{2\sqrt{h}} (h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd}) \quad (73)$$

and made use of the abbreviations $\mathcal{M} = 1/32\pi G$ and $V[h_{ab}] = -2\sqrt{h}(R - 2\Lambda)$. The classical theory is completely described by a set of constraints: the well-known Hamiltonian and diffeomorphism constraints, and a couple of extra constraints arising from boundary terms.

Quantization is now performed in the standard formal manner by replacing all momenta with \hbar/i times (functional) derivatives and implementing all constraints by acting on wave functionals $\Psi[h_{ab}(x), \phi(x), \theta_m(x); \tau, \tau_a, \tau_{ab}, \theta)$. At this point one normally has to rely on a particular factor ordering. But since later on only the semiclassical behaviour is considered explicitly, we do not need to fix this ambiguity. The constraint equations read:

$$\mathcal{H}_{inner}\Psi = 0, \quad (74)$$

$$\mathcal{C}_{outer}\Psi = 0. \quad (75)$$

Note that in contrast to the standard discussion of the Wheeler-DeWitt approach one now has also to take into account the outer constraints (75).

To define thermodynamical properties one would like to generalize (56) to

$$\kappa\delta\frac{\partial\Psi}{\partial\theta} + \alpha_0\delta\frac{\partial\Psi}{\partial\tau} + \alpha_a\delta\frac{\partial\Psi}{\partial\tau_a} - \frac{1}{2}\beta_{ab}\delta\frac{\partial\Psi}{\partial\tau_{ab}} + \int_J d^2x N_m\delta\frac{\delta\Psi}{\delta\theta_m} = 0. \quad (76)$$

Using the constraints (75), this would lead to the classical result. But as was discussed in section 4, the validity of equation (56) is based on stationary spacetimes (as well as on certain classes of perturbation). In the classical theory stationarity is provided by the existence of timelike Killing vector fields. But what this classical property of black hole spacetimes means in the quantized theory is quite unclear. Therefore we shall discuss in the following a semiclassical approximation, where one can at least approximately rely on such classical features.

7 Semiclassical Approximation

Starting point for the semiclassical approximation is the distinction between ‘nearly’ classical and ‘fully’ quantum theoretical quantities. This can be compared with a similar situation in molecular physics, where a dynamical distinction between the heavy nuclei and the light electron can be profitably used to develop the Born-Oppenheimer approximation. For our case we assume gravity to take over the role of the slowly moving nuclei, while the matter field takes over the role of the fast moving electrons. As is shown in [13], this qualitative difference can

be formally introduced into the approximation scheme by choosing a WKB-type ansatz for the state functional

$$\Psi = \exp\left(\frac{i}{\hbar}S\right) \quad (77)$$

and by expansion of S in powers of \mathcal{M} ,

$$S = \mathcal{M}S_0 + S_1 + \mathcal{M}^{-1}S_2 + \dots \quad (78)$$

In the highest order (\mathcal{M}^2) one finds that S_0 is independent of ϕ , i.e. independent of the matter fields. In the next order (\mathcal{M}^1) one gets the Hamilton-Jacobi equation for the gravitational field

$$\frac{1}{2}G_{abcd}\frac{\delta S_0}{\delta h_{ab}}\frac{\delta S_0}{\delta h_{cd}} + V[h_{ab}] = 0. \quad (79)$$

This equation (together with the corresponding diffeomorphism equation) is equivalent to all ten of Einstein's field equations (here in vacuum) and therefore describes the classical solutions of general relativity – in our case the classical black hole solutions. The solution of this equation can also be used to discuss the meaning of classically allowed and forbidden regions [8].

In contrast to the standard discussion one finds in addition to the Hamilton-Jacobi equation (79) the constraints:

$$\frac{\partial S_0}{\partial \theta} - \frac{A(r_+)}{8\pi G} = 0, \quad (80)$$

$$\frac{\delta S_0}{\delta \theta_m} + 2r_n p_\sigma^{mn} = 0, \quad (81)$$

$$\frac{\partial S_0}{\partial \tau_a} + 2 \int_{B_t} d^2x [r_b p_\sigma^{ab}] = 0, \quad (82)$$

$$-\frac{1}{2} \frac{\partial S_0}{\partial \tau_{ab}} - \int_{B_t} d^2x r_c [x^a p_\sigma^{bc} - x^b p_\sigma^{ac}] = 0, \quad (83)$$

$$\frac{\partial S_0}{\partial \tau_0} + M = 0. \quad (84)$$

They are easily solved by a separation ansatz

$$S_0[h_{ab}(x), \theta_m(\tilde{x}); \tau_0, \tau_a, \tau_{ab}, \theta] = S_0[h_{ab}] + S_0[\theta_m] + S_0(\tau_0) + S_0(\tau_a) + S_0(\tau_{ab}). \quad (85)$$

As a consequence of the discussion above, the boundary contributions to the Hamilton-Jacobi functional are connected with the thermodynamical properties of black holes. Using equation (56) one finds

$$\kappa \delta \frac{\partial S_0}{\partial \theta} + \alpha_0 \delta \frac{\partial S_0}{\partial \tau_0} + \alpha_a \delta \frac{\partial S_0}{\partial \tau_a} - \frac{1}{2} \beta_{ab} \delta \frac{\partial S_0}{\partial \tau_{ab}} + \int_J N_m \delta \frac{\partial S_0}{\partial \theta_m} d^2x = 0. \quad (86)$$

Note that this relation only holds if the classical black hole spacetime described by S_0 possesses a timelike Killing vector field. For equilibrium states, such a timelike Killing vector field must exist for every order of our approximation scheme. Thus, the generalization of (56) from order to order should lead to the corresponding thermodynamical description.

In the next order (\mathcal{M}^0) one finds by introducing the wave functional

$$\chi = D[h_{ab}] \exp(iS_1/\hbar), \quad (87)$$

the local functional Schrödinger equation

$$i\hbar G_{abcd} \frac{\delta S_0}{\delta h_{ab}} \frac{\delta \chi}{\delta h_{cd}} \equiv i\hbar \frac{\delta \chi}{\delta \mathcal{T}} = \mathcal{H}_\phi \chi \quad (88)$$

for quantum fields propagating on the classical spacetime described by S_0 . The functional D is thereby chosen in such a way that it obeys the standard WKB prefactor equation [13]. Taking into account the boundary terms we find that S_1 is independent of all gravitational boundary variables, while surface terms of the matter fields would lead to an additional contribution. If there are no additional surface terms of the quantum fields, the thermodynamical properties of the black hole will remain unchanged at first glance.

Until now there is one weak point in our discussion – our fixation procedure of the hypersurface at the bifurcation point presupposes the knowledge of its position. This horizon position r_+ we had to put in by hand. This intervention by hand also fixes the thermodynamical properties of our theory. One thus has to find a way to define the position of the horizon in every order of approximation. In the order \mathcal{M}^0 this can be achieved with the semiclassical Hamilton-Jacobi equation discussed in [13]. To define this ‘corrected’ equation one has to decompose the functional χ into

$$\chi \equiv C \exp(i\vartheta) \quad (89)$$

and write the Hamilton-Jacobi equation (79) up to order \mathcal{M} in the form

$$\frac{1}{2\mathcal{M}} G_{abcd} \left(\mathcal{M} \frac{\delta S_0}{\delta h_{ab}} + \frac{\delta \vartheta}{\delta h_{ab}} \right) \left(\mathcal{M} \frac{\delta S_0}{\delta h_{cd}} + \frac{\delta \vartheta}{\delta h_{cd}} \right) + \mathcal{M} V[h_{ab}] - \frac{\delta \vartheta}{\delta \mathcal{T}} + \mathcal{O}(\mathcal{M}^{-1}) = 0. \quad (90)$$

Next we take the expectation value with respect to χ and use the relation

$$\langle \chi | \frac{\delta \vartheta}{\delta \mathcal{T}} \chi \rangle = - \langle \chi | \mathcal{H}_\phi \chi \rangle \quad (91)$$

to write

$$\frac{1}{2\mathcal{M}} G_{abcd} \Pi^{ab} \Pi^{cd} + \mathcal{M} V[h_{ab}] + \langle \chi | \mathcal{H}_\phi \chi \rangle + \mathcal{O}(\mathcal{M}^{-1}) = 0, \quad (92)$$

where we have introduced the notation

$$\Pi^{ab} \equiv \mathcal{M} \frac{\delta S_0}{\delta h_{ab}} + \langle \chi | \frac{\delta \vartheta}{\delta h_{ab}} \chi \rangle. \quad (93)$$

As is argued in [13], one has in this order of approximation to interpret the momenta Π^{ab} as the geometrodynamical momenta. Therefore the position of the horizon has now to be derived from the ‘back reaction corrected’ Hamilton-Jacobi functional \tilde{S}_0 given by

$$\frac{1}{2\mathcal{M}} G_{abcd} \frac{\delta \tilde{S}_0}{\delta h_{ab}} \frac{\delta \tilde{S}_0}{\delta h_{cd}} + \mathcal{M} V[h_{ab}] + \langle \chi | \mathcal{H}_\phi \chi \rangle + \mathcal{O}(\mathcal{M}^{-1}) = 0. \quad (94)$$

Note that in the above investigation we assumed the wave functional to be in a particular WKB-state. But since the fundamental equations are linear, one would expect arbitrary superpositions of such states to occur. It can easily be seen that such superpositions are only possible for ‘inner’ states (solutions of the inner constraints). The outer constraints do not allow for superpositions and must be interpreted as describing superselection rules (fixing mass, angular momentum, etc.). Nevertheless, since the wave functional at this order of approximation reads

$$\Psi \approx e^{iS_0} \chi, \quad (95)$$

one could, for example, also discuss the superposition

$$\Psi \approx \left(e^{iS_0[h_{ab}]} \chi + e^{-iS_0[h_{ab}]} \bar{\chi} \right) e^{iS_0(\theta, \tau_a, \dots)}. \quad (96)$$

This state could naively be called a “superposition of a black hole with a white hole”. As was shown in [25] for a two-dimensional dilaton gravity model, such superpositions are suppressed by decoherence. Decoherence is caused by the presence of a huge number of irrelevant degrees of freedom. In [25] it is argued that even the correlation between Hawking radiation (viewed there as the irrelevant degrees of freedom) and the black hole can lead to decoherence. Thus in this sense the various semiclassical components should become dynamically independent by radiation of the black hole itself. This would give additional support to the above consideration of particular WKB-states.

After this very general discussion one would now like to consider the particular example of 3+1 dimensional black hole solutions. But since no consistent derivation of back reaction exists in four dimensions, one has to look out for a more simple model. As we will see in the next section such a model exists in 2+1 dimensional gravity.

8 BTZ Black Holes as an Example

8.1 Classical Description

Some years ago a black hole solution in $2 + 1$ -dimensional gravity was discovered [26]. Since these BTZ black holes are embedded in a ‘well-understood’ 3-dimensional gravity theory, they have provoked particular interest in the last few years [27]. Recently attempts were made to study back reaction on their geometry by matter fields [28]. An exact solution of the corresponding semiclassical Einstein equation was found for the special case of conformal fields [21]. For this reason, $2 + 1$ gravity provides a good model for our purposes. To connect this model with our discussion above, we shall first of all construct the corresponding Hamilton-Jacobi functional for the classical, matter free theory.

We start from the action

$$I = \frac{1}{2} \int d^3x \sqrt{-g} \left[\frac{\mathcal{R} + 2l^{-2}}{\lambda} \right] + I_b \quad (97)$$

without any matter field. Note, that $-l^{-2}$ denotes the cosmological constant and that I_b stands for an appropriate boundary term. Note also that in this subsection we shall for simplicity use units in which $\lambda = 8\pi G = 1$. Given the action, we have to perform the ADM decomposition next. Let us therefore denote the induced metric on the two-dimensional hypersurfaces Σ by

$$d\sigma^2 = L^2 dr^2 + 2Q^2 dr d\phi + R^2 d\phi^2. \quad (98)$$

Restriction to axisymmetric solutions then leads after some calculations to the constraints

$$\tilde{\mathcal{H}} \equiv -\frac{\Pi_L \Pi_R}{LR} + \frac{1}{4Q^2} \Pi_Q^2 - \frac{1}{(\det \sigma)^2} \left[Q^6 \frac{d}{dr} \left(\frac{RR'}{Q^2} \right) - R^3 L^3 \frac{d}{dr} \left(\frac{R'}{L} \right) \right] - l^{-2} \approx 0, \quad (99)$$

$$\tilde{\mathcal{D}}^r \equiv \frac{1}{2} Q' \Pi_Q - \frac{1}{2} Q \Pi_Q' - L \Pi_L' + R' \Pi_R \approx 0, \quad (100)$$

$$\tilde{\mathcal{D}}^\phi \equiv \left(\frac{Q^2}{L} \Pi_L \right)' + \left(\frac{R^2}{2Q} \Pi_Q \right)' \approx 0. \quad (101)$$

The last constraint can immediately be integrated and gives the relation

$$\Pi_Q = -\frac{2Q^3}{LR^2} \Pi_L + \frac{Q}{R^2} J, \quad (102)$$

where $J/2$ is the corresponding integration constant. Inserting this relation into the Hamilton constraint (99) and demanding $Q \equiv 0$ the above constraint system reduces to

$$\mathcal{H} \equiv -\Pi_L \Pi_R + \frac{L}{4R^3} J^2 + \frac{R''}{L} - \frac{L'R'}{L^2} - LRl^{-2} \approx 0, \quad (103)$$

$$\mathcal{D}^r \equiv -L \Pi_L' + R' \Pi_R \approx 0. \quad (104)$$

Note that the restriction $Q \equiv 0$, which leads to a simplification of the constraint system, can only be done after inserting (102) into (99). For the new constraints one can construct the corresponding Hamilton-Jacobi functional in the way described in [8]. It reads

$$S_0 = \pm LB \mp R' \ln \left[2R' \left(B + \frac{R'}{L} \right) \right], \quad (105)$$

where

$$B = \left(\frac{R'^2}{L^2} + M - \frac{J^2}{4R^2} - \frac{R^2}{l^2} \right)^{1/2}, \quad (106)$$

and where the integration constant M can later be identified with the ADM-mass. Note that for $\lambda = 1$ the ADM-mass M is dimensionless. Remarkably, the ADM-mass can on-shell be read off from the functional

$$M(r) = \Pi_L^2 + \frac{J^2}{4R^2} - \left(\frac{R'}{L} \right)^2 + \frac{R^2}{l^2}. \quad (107)$$

This functional is an exact analogon to expressions found for 1+1 dimensional dilaton gravity and for the spherically symmetric sector of Einstein gravity [8]. It can be used to determine the position of the horizon for arbitrary parametrizations. One obtains the condition [8]

$$\left(\frac{R'}{L} \right)^2 - \Pi_L^2 = 0. \quad (108)$$

Once the Hamilton-Jacobi functional is given, it is straightforward to deduce an explicit expression for the metric. Choosing $R(r) = r$ one recovers the BTZ black hole solution [26]

$$ds^2 = -F dt^2 + F^{-1} dr^2 + r^2 (N^\phi dt + d\phi)^2 \quad (109)$$

with

$$F = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}, \quad (110)$$

$$N^\phi = -\frac{J}{2r^2}. \quad (111)$$

The black hole horizon following from (108) or (110) reads

$$r_\pm = l \left[\frac{M}{2} \left(1 \pm \left[1 - \left(\frac{J}{Ml} \right)^2 \right]^{1/2} \right) \right]^{1/2}. \quad (112)$$

The entropy is determined by the boundary term which here reads

$$I_b = -2\pi \int dt \left[\pi_\theta \dot{\theta} + P^0 \dot{\tau}_0 + P^\omega \dot{\tau}_\omega - \kappa \mathcal{C}_\theta - N \mathcal{C}_0 - N^\phi \mathcal{C}_\omega \right] \quad (113)$$

with

$$\mathcal{C}_\theta \equiv \pi_\theta - 2r_+ \approx 0, \quad (114)$$

$$\mathcal{C}_0 \equiv P^0 + M \approx 0, \quad (115)$$

$$\mathcal{C}_j \equiv -P^\omega + J \approx 0. \quad (116)$$

Integration of t up to $\beta = 2\pi/\kappa$ leads then to an entropy

$$S = 4\pi^2 r_+. \quad (117)$$

Thus, for $J = 0$ the entropy is explicitly given by $S = 4\pi^2 \sqrt{M}l$. From the standard relation $T = -(g'_{tt})_{r_+}/4\pi$ ($g_{tt}g_{rr} = -1$) the temperature is ($J = 0$):

$$T = \frac{\sqrt{M}}{2\pi l}. \quad (118)$$

Note that the Smarr formula for the BTZ solutions reads $2M = TS$, which follows immediately from the Euler theorem for homogeneous functions.

8.2 Conformally coupled Matter Fields

To introduce (conformally coupled) matter fields we consider the following action:

$$I = \frac{1}{2} \int d^3x \sqrt{-g} \left[\frac{R + 2l^{-2}}{\lambda} - g^{\mu\nu} \nabla_\mu \psi \nabla_\nu \psi - \frac{R}{8} \psi^2 \right]. \quad (119)$$

As in the last section one could start to perform the ADM decomposition. But this leads to a rather complicated system of constraints, for which we weren't able to construct the Hamilton-Jacobi functional. On the other hand for a consistent choice of ψ [29] one finds in the spherically symmetric case the solutions

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2 d\theta^2, \quad (120)$$

with

$$F(r) = \frac{1}{l^2} \left[r^2 - 3B^2 - \frac{2B^3}{r} \right], \quad (121)$$

$$\psi(r) = \sqrt{\frac{8B}{\lambda(r+B)}}. \quad (122)$$

B is here an arbitrary constant. The horizon of these solutions is at

$$r = 2B \equiv r_+, \quad (123)$$

and the thermodynamical quantities are [29]

$$T = \frac{9r_+}{16\pi l^2}, \quad S = \frac{8\pi^2 r_+}{3\lambda}. \quad (124)$$

In [29] it is noted ‘that the entropy differs by a factor 2/3 from the “area law” $4\pi^2 r_+/\lambda$ ’. Our above consideration now suggests that this deviation from the “area law” is a consequence of the conformal coupling of the matter field.⁵ In this case one has to take into account an additional boundary term, which leads to

$$S = \frac{4\pi^2 r_+}{\lambda} \left(1 - \frac{\psi(r_+)^2}{8} \lambda \right) = \frac{8\pi^2 r_+}{3\lambda}. \quad (125)$$

Note that in our notation of the last section we have $\lambda = 1$, whereas in [29] one has $\lambda = 1/8\pi$. Note also that again the Smarr formula $2M = TS$ holds.

8.3 Semiclassical Description and Back Reaction

To introduce back reaction we study the matter action (119) with $\lambda = 1$. The corresponding renormalized expectation value of the stress-tensor is calculated in [21] and reads for the so-called transparent boundary condition (in the case $J = 0$)

$$\langle T_\mu^\nu \rangle = \frac{l_p M^{3/2} \mathcal{A}(M)}{R^3} \text{diag}(1, 1, -2), \quad (126)$$

with $l_p = \lambda/8\pi$ and where

$$\mathcal{A}(M) \equiv \frac{1}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{\cosh 2n\pi\sqrt{M} + 3}{(\cosh 2n\pi\sqrt{M} - 1)^{3/2}}. \quad (127)$$

This leads to a contribution to the Hamilton constraint (103). This additional contribution can immediately be incorporated if one recalls that the Hamilton constraint can be read off from the projection of the Einstein equation onto the normal to Σ . One gets

$$\bar{\mathcal{H}} \equiv -\bar{\Pi}_L \bar{\Pi}_R + \frac{R''}{L} - \frac{L' R'}{L^2} - L R l^{-2} - \frac{L}{R^2} l_p M^{3/2} \mathcal{A}(M) \approx 0, \quad (128)$$

$$\bar{\mathcal{D}}^r \equiv -L \bar{\Pi}'_L + R' \bar{\Pi}_R \approx 0, \quad (129)$$

where the use of ‘bared’ quantities is meant to distinguish between the ‘uncorrected’ (103,104) and the back reaction-corrected treatment. The corresponding Hamilton-Jacobi solution now reads

$$\bar{S}_0 = \pm L \bar{B} \mp R' \ln \left[2R' \left(\bar{B} + \frac{R'}{L} \right) \right] \quad (130)$$

where

$$\bar{B} = \left(\frac{R'^2}{L^2} + M - \frac{R^2}{l^2} + \frac{2l_p}{R} M^{3/2} \mathcal{A}(M) \right)^{1/2}. \quad (131)$$

⁵ The same conclusion follows from the Noether charge consideration of black hole entropy [16, 30, 31]

Interestingly enough the value of the ADM mass is not changed by the back reaction of the matter field [21]. From (131) we find for the horizon

$$\bar{r}_+ = \sqrt{M}l \left(1 + \frac{l_p}{l} \mathcal{A}(M) \right), \quad (132)$$

where the second term is a l_p correction in the sense of the above discussed semiclassical approximation scheme. Thus the entropy reads

$$\bar{S} = 4\pi^2 \bar{r}_+ \left(1 - \frac{\langle \psi^2 \rangle}{8} \right). \quad (133)$$

Note that we have introduced the boundary term of the conformal coupling by replacing ψ^2 with $\langle \psi^2 \rangle$, since otherwise this term would lead to an inconsistency with the quantum theoretical description of the matter field⁶. The temperature for the black hole solution (130) is, up to order $O(l_p^2)$,

$$\bar{T} = \frac{\sqrt{M}}{2\pi l} \left(1 + 2\frac{l_p}{l} \mathcal{A}(M) \right). \quad (134)$$

Since $\mathcal{A}(M)$ is a complicated expression in M , the entropy \bar{S} is no longer a homogeneous function in M . Thus one cannot expect the Smarr formula $2M = TS$ to hold anymore. On the other hand, the first thermodynamical relation $dM = \bar{T}d\bar{S}$ must be valid by construction. This leads to the consistency condition $d\bar{S}/dM = 1/\bar{T}$, which determines the value of $\langle \psi^2 \rangle$ at the horizon. Evaluation of $\bar{S} = \int dMT^{-1}$ up to order $\mathcal{O}(l_p^2)$ and comparison with the expression (133) for \bar{S} in the case of $M \gg 1$, i.e. $\mathcal{A}(M) \approx \exp(-\pi\sqrt{M})/2$, gives

$$\langle \psi^2 \rangle_{r_+} \approx 4\frac{l_p}{l} \exp(-\pi\sqrt{M}) \left(1 - \frac{2}{\pi\sqrt{M}l} \right). \quad (135)$$

Thus the entropy reads

$$\bar{S} = 4\pi^2 \sqrt{M}l + 4\pi \frac{l_p}{l} \exp(-\pi\sqrt{M}) + \mathcal{O}(l_p^2) \quad (136)$$

Recall that ‘per definition’ our semiclassical description is only valid if gravity behaves classically. This naturally implies the considered limit $M \gg 1$.

Using on the other hand the relation between the two point function and the appropriate Green function [28, 32], $\langle \psi^2 \rangle = \frac{1}{2} \lim_{x \rightarrow x'} G_{reg}(x, x')$, one receives for the considered transparent boundary condition

$$\langle \psi^2 \rangle = \frac{1}{4\sqrt{2\pi}l} \frac{r_+}{r} \sum_{n \neq 0} \left\{ \cosh(2\pi\sqrt{M}n) - 1 \right\}^{-1/2} \quad (137)$$

⁶The same ansatz was suggested recently from a different point of view [33].

For $M \gg 1$ only the $n = \pm 1$ contributions of the sum have to be considered. One gets

$$\langle \psi^2 \rangle_{r_+} \approx \frac{1}{2\pi} \frac{1}{l} \exp(-\pi\sqrt{M}) = 4 \frac{l_p}{l} \exp(-\pi\sqrt{M}), \quad (138)$$

where we have used $l_p = \lambda/8\pi = 1/8\pi$. In the case of $M \gg 1$ this expression equals (135). Thus consistency of the formula (133) for $M \gg 1$ is shown.

Let us emphasize that our result for the entropy differs from the result found in [21], where no contribution from the non-minimally coupled matter field was respected. But this contribution has to be taken into account since otherwise the relation $dM = TdS$ does not hold.

9 Final Remarks

In this paper we have discussed the canonical quantization of spacetimes with boundaries. A key role in our investigation is played by additional degrees of freedom arising from surface terms. We have shown how these additional degrees of freedom are connected with standard relations of black hole thermodynamics. Moreover, we have given a derivation of the Nernst theorem. In the quantum theory the thermodynamical properties are rediscovered in the first order of a WKB-like approximation scheme. Going beyond the first order of this approximation leads to back reaction effects. We have shown how these back reaction effects alter the thermodynamical properties. We in particular have discussed how our treatment works in the case of the BTZ solution in 2+1 dimensional gravity. Moreover, this example has provided useful insights into the correlation between boundary terms and entropy.

Let us note, that another approach which is in some aspect similar in the interpretation of entropy is the Noether charge derivation [16, 30, 31]. From the Noether charge approach one would, for example deduce similar conclusions we have drawn for the particular example of the BTZ model. Since the Noether charge interpretation of black hole thermodynamics is valid for all diffeomorphism invariant theories, we expect the same for our consideration, although no generalization of the WKB approximation for arbitrary diffeomorphism invariant theories exists. Whether such a generalization also leads to an analog of the semiclassical Hamilton-Jacobi equation is an open issue. If this is the case one could, of course, capture back reaction in the above described way.

Note also that for the discussion of back reaction from Hawking radiation one has to generalize the given thermodynamical description to at least quasi stationary spacetimes. But even in the case of radiating black holes, the stationarity assumption should be valid as long as the black hole mass is much bigger than the Planck mass.

As it was recently pronounced [22] it is somehow ironic that of all things, new developments in string theory (which seem to provide a statistical explanation

for black hole thermodynamics) have led to a contrary result for the entropy of extremal black holes. In these developments the statistical interpretation was achieved by counting so called BPS states in the weakly coupled string theory. This counting was explicitly done for extremal and nearly extremal black holes and led to the Bekenstein-Hawking formula $S = A/4G$, whereas for non extremal black holes a consistent counting is still elusive. Until now a satisfying explanation for the differing conclusions between the canonical theory of gravity and string theory has not been given. Nevertheless, first attempts were made and they seem to suggest that modification from string scales may play an important role even if the curvature of the Euclidean solution is everywhere small (in this case string corrections should be negligible) [22]. It would be interesting if such string modifications could be connected with observations made by Davies [34] who found a phase transition for Kerr black holes at $J/M \simeq 0.68$ and Reissner-Nordström black holes at $q/M \simeq 0.86$. If such a correlation could be derived it would be a beautiful improvement for the acting in unison of string theory and general relativity. Of course, at the same time one again would be faced by the Nernst theorem, which then would be manifestly broken in string theory. But since new investigations seem to suggest that the Nernst theorem is also violated for a ideal boson gas which is confined on a circular string [35], this may indicate that the Nernst formulation of the third law of thermodynamics will lose its fundamental role in string theory.

However, since black hole thermodynamics seems to be the only vague glimpse we get from the quantum theory of gravity, contrary results between different approaches are of inestimable value and should be viewed as a chance to stake off the appropriate camp for quantum gravity.

Acknowledgement

It is a pleasure to thank Domenico Giulini and Claus Kiefer for very helpful and stimulating discussions. I would especially like to thank Claus Kiefer for critically reading the manuscript and making valuable comments. Financial supports from the *Graduiertenkolleg Nichtlineare Differentialgleichungen der Albert-Ludwigs-Universität Freiburg* are acknowledged.

References

- [1] B.S. DeWitt, Phys. Rev. **160**, 1113 (1967).
- [2] T. Regge and C. Teitelboim, Ann. Phys. (N.Y.) **88**, 286 (1974).
- [3] R. Beig and N. ó Murchadha, Ann. Phys. (N.Y.) **174**, 463 (1987).
- [4] K.V. Kuchař, Phys. Rev. D **50**, 3961 (1994).
- [5] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15**, 2752 (1977); S.W. Hawking, in *General Relativity*, edited by S.W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979).
- [6] S. Carlip and C. Teitelboim, Class. Quantum Grav. **12**, 1699 (1995).
- [7] C. Teitelboim, Phys. Rev. D **51**, 4315 (1995).
- [8] T. Brotz and C. Kiefer, Phys. Rev. D **55**, 2186 (1997).
- [9] J. D. Brown, Report-No. gr-qc/9704071.
- [10] S. Bose, L. Parker, and Y. Peleg, Phys. Rev. D **56**, 987 (1997).
- [11] D. Sudarsky and R.M. Wald, Phys. Rev. D **46**, 1453 (1992).
- [12] S. Carlip, Phys. Rev. D **55**, 878 (1997).
- [13] C. Kiefer, in *Canonical Gravity: From Classical to Quantum*, edited by J. Ehlers and H. Friedrich (Springer, Berlin, 1994).
- [14] S.W. Hawking and C.J. Hunter, Class. Quantum Grav. **13**, 2735 (1996).
- [15] G. Hayward, Phys. Rev. D **47**, 3275 (1993).
- [16] R.M. Wald, *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics* (University of Chicago Press, Chicago, 1994).
- [17] R.M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).
- [18] G. W. Gibbons and R. E. Kallosh, Phys. Rev. D **51**, 2839 (1995); S. W. Hawking, G. T. Horowitz, and S. F. Ross, *ibid.* **51**, 4302 (1995).
- [19] S.W. Hawking and G.T. Horowitz, Class. Quantum Grav. **13**, 1487 (1996).
- [20] E.A. Martinez, Phys. Rev D **51**, 5732 (1995).
- [21] C. Martínez and J. Zanelli, Phys. Rev. D **55**, 3642 (1997).
- [22] G.T. Horowitz, Report-No. gr-qc/9604051.

- [23] J.D. Brown, Phys. Rev. D **52**, 7011 (1995).
- [24] S.W. Hawking, Commun. Math. Phys. **43**, 199 (1975).
- [25] J.-G. Demers and C. Kiefer, Phys. Rev. D **53**, 7050 (1996).
- [26] M. Banados, C. Teitelboim, and J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992).
- [27] S. Carlip, Class. Quantum Grav. **12**, 2853 (1995).
- [28] A. R. Steif, Phys. Rev. D **49**, R585 (1994).
- [29] C. Martínez and J. Zanelli, Phys. Rev. D **54**, 3830 (1996).
- [30] R.M. Wald, Phys. Rev. D **48**, R3427 (1993).
- [31] V. Iyer and R.M. Wald, Phys. Rev. D **50**, 846 (1994).
- [32] G. Lifschytz and M. Ortiz, Phys. Rev. d **49**, 1929 (1994).
- [33] V. P. Frolov, D. V. Fursaev, and A. I. Zelnikov, Nucl. Phys. B (Proc. Suppl.) **57** 192 (1997).
- [34] P.C.W. Davies, Proc. R. Soc. A **353**, 499 (1977).
- [35] R.M. Wald, *The Nernst Theorem and Black Hole Thermodynamics*, Report-No. gr-qc/9704008.